INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 8, 81–90 (1988)

A NUMERICAL METHOD FOR ACOUSTIC OSCILLATIONS IN TUBES*

JOHN M. GARY

Scientific Computing Division, 713 National Bureau of Standards, 325 S. Broadway, Boulder, CO 80303. U.S.A.

SUMMARY

A numerical method to obtain the neutral curve for the onset of acoustic oscillations in a helium-filled tube is described. Such oscillations can cause a serious heat loss in the plumbing associated with liquid helium dewars. The problem is modelled by a second-order, ordinary differential eigenvalue problem for the pressure perturbation. The numerical method to find the eigenvalues and track the resulting points along the neutral curve is tailored to this problem. The results show that a tube with a uniform temperature gradient along it is much more stable than one where the temperature suddenly jumps from the cold to the hot value in the middle of the tube.

KEY WORDS Hydrodynamic stability Acoustic waves Cryogenic oscillations Taconis vibration Sondhauss tube Continuation

INTRODUCTION

A long tube filled with helium which has an open end at a very low temperature and a closed end at room temperature may develop a spontaneous acoustic oscillation. The oscillation results from the temperature difference between the ends of the tube. Such oscillations often occur in plumbing associated with liquid helium dewars, and the resultant heat leak can be very large. Methods exist for damping these oscillations when they do occur.[1] However, predictions for their existence are often unreliable, since the existing model[2] has an unrealistic assumption. The study of acoustic waves in long tubes goes back to Helmholtz in 1863. Rott, in a series of papers, developed a model for the helium tube instability and obtained various solutions of this model.[2] Most of these solutions were obtained from an asymtotic expansion based on a piecewise constant mean temperature along the tube. In these cases the temperature is assumed to have a discontinuous jump from the cold end value to the hot end value at some interior point along the tube.

The model for this problem is a second-order, ordinary differential equation for the pressure perturbation as a function of the distance along the tube. This equation yields an eigenvalue problem for the neutral stability curve. This paper describes a numerical method to solve this differential eigenvalue problem. A shooting method for the differential equation is used along with a non-linear algebraic equation solver to find the eigenvalues. Both these methods use subroutine packages available on many computing systems. To obtain a neutral curve by this technique, an accurate method to extrapolate a new point on the neutral curve from previously obtained values seems to be required. This is because some regions of the neutral curve require a very accurate

* Contribution of the National Bureau of Standards, not subject to copyright in the United States.

0271–2091/88/010081–10$05.00

Received 23 September 1986
Revised 12 March 1987

initial guess for the eigenvalue; otherwise the equation solver will not converge. The extrapolation method is described in some detail, since it is crucial for convergence.

If the piecewise constant temperature profile is used, then an analytic solution of the differential equation can be obtained. Then the differential eigenvalue problem reduces to an algebraic equation. This is the equation which Rott solved by an asymptotic approximation. A direct solution of this problem, which uses the same algebraic equation solver and the same extrapolation as used for the differential equation, is described.

This direct solution for the step function temperature agrees very well with Rott's solution. In addition, solutions for several smoothly varying temperature profiles were obtained. As these smooth curves become steeper and approach the step function, the associated neutral curves converge to that for the step function.

THE MATHEMATICAL MODEL

The model is developed in a paper by Rott.[2] It is based on the Navier–Stokes equations, for a viscous, conducting fluid in a cylindrical tube whose radius is assumed to be much smaller than its length. The equations are linearized about a mean state and a harmonic time variation, $e^{i\omega t}$, is assumed. The model is based on the Navier–Stokes equations for viscous flow in a cylinder. It assumes no angular dependence and also assumes that the pressure of the gas is independent of the radius. The radial dependence can be eliminated from the model by solving a differential equation in the radial direction. The result is a differential equation in the axial, or x, direction which involves Bessel functions from the solution in the radial direction.

It is the following differential equation for the complex pressure amplitude $p(x)$ along the tube, $0 \leqslant x \leqslant L$. It is equation (19) in Rott's[2] paper. The cold end is at $x = 0$.

$$A(x)p'' + B(x)p' + C(x)p = 0. \tag{1}$$

The boundary conditions are

$$p(0) = 0, \quad p'(L) = 0. \tag{2}$$

The coefficient functions are

$$\begin{aligned}
A(x) &= a(x)^2[1 - f(x)]/\omega^2,\\
B(x) &= a(x)^2[f(x) - f^*(x)]\theta/(\omega^2(1-\sigma)) + [a(x)^2]'/\omega^2\\
&\quad - a(x)^2 f'(x)/\omega^2 - a(x)^2\theta f(x)/\omega^2,\\
C(x) &= 1 + (\gamma - 1)f^*(x),\\
f(x) &= 2J_1(z)/zJ_0(z),\\
f^*(x) &= 2J_1(\sqrt{\sigma}z)/\sqrt{\sigma}zJ_0(z),\\
z &= i^{3/2}Y_c[T_m(0)/T_m(x)]^{(1+\beta)/2},\\
\theta &= T'_m(x)/T_m(x),\\
Y_c &= [\omega/\nu(0)]^{1/2}r_0.
\end{aligned}$$

Here $J_0(z)$ and $J_1(z)$ denote the Bessel functions of order zero and one, $a(x)$ is the sound speed, $T_m(x)$ is the temperature profile, $\nu(0)$ is the kinematic viscosity at the cold temperature and r_0 is the tube radius. A more complete description of these parameters is given in Appendix 1.

The differential equation (1) can be written as a system of four real first-order equations whose unknown functions are

$$U_1(x) = \mathrm{Re}\,[p(x)],\quad U_2(x) = \mathrm{Im}\,[p(x)],\quad U_3(x) = \mathrm{Re}\,[p'(x)],\quad U_4(x) = \mathrm{Im}\,[p'(x)].$$

This is an eigenvalue problem. For a fixed mean temperature the parameters ω and Y_c must be chosen so that a non-trivial solution of equations (1) and (2) is possible. The solution is obtained by a shooting method. The arbitrary complex multiplier in the solution $p(x)$ is eliminated by the requirement that $p'(0) = 1$ or $U_3(0) = 1$, $U_4(0) = 0$. The boundary condition at $x = 0$ gives $U_1(0) = U_2(0) = 0$, and therefore a full set of initial conditions for the integration of (1) is obtained. Then the parameters ω and Y_c must be chosen so that the boundary condition at $x = L$ is satisfied; that is, $U_3(L) = U_4(L) = 0$. This can be regarded as a system of two equations in two unknowns; namely,

$$U_3(L, \omega, Y_c) = 0, \quad U_4(L, \omega, Y_c) = 0. \tag{3}$$

To obtain a neutral curve, the solution for (ω, Y_c) must be obtained for a family of temperature functions. This family is parametrized by the ratio of the temperatures at the hot and cold ends; that is, $\alpha = T_M(L)/T_m(O)$. Then $T_m = T_m(x,\alpha)$, where $T_m(L,\alpha)/T_m(0,\alpha) = \alpha$. The result depends on the shape of the curve as well as the ratio α. The family of temperature functions is described in Appendix I. The stability region for the oscillation is then determined by the function $Y_c(\alpha)$ or $\omega(\alpha)$. These are the values which give neutral stability (that is, the wave number ω is real) for a fixed α. The stability curve also depends on the shape of the temperature $T_m(x, \alpha)$. The shape of the functions used in this paper depends on two additional parameters, as described in Appendix II. These parameters can be chosen so that the temperature profile approaches a step function.

The numerical solution for the step function temperature profile is essentially the same, except the differential equation can be solved analytically. As shown in Rott's paper,[2] the eigenvalue problem is obtained from an interface condition which must be satisfied at the point where the jump occurs in the temperature. Using the notation from Rott's paper, this condition is

$$\frac{G_c}{k_c}\cot(k_c j) = \frac{G_n}{k_n}\tan[k_n(L - j)], \tag{4}$$

where the jump occurs at $x = j$ and the remaining variables are defined below:

$$\begin{aligned}
G(x) &= g_1(x)E(x),\\
g_1(x) &= 1 + (\gamma - 1)f^*(x),\\
E(x) &= \exp\left(\int_0^x (g_3/g_2)\,\mathrm{d}x\right),\\
g_2(x) &= a^2(1 - f)/\omega^2,\\
g_3(x) &= a^2(f - f^*)\theta/\omega^2(1 - \sigma),\\
k(x) &= (g_1/g_2)^{1/2} = \omega\{[1 + (\gamma - 1)f^*]/(1 - f)\}^{1/2}/a.
\end{aligned}$$

The subscripts c and n on G and k refer to the cold and hot end temperatures respectively. If the temperature function is monotone—that is, $T'_m > 0$— then the integration used to define E can be transformed to an integration over T. Thus

$$E(x) = \exp\left(\int_{T_c}^{T(x)} \{[f(T) - f^*(T)]/(1 - \sigma)[1 - f(T)]T\}\,\mathrm{d}T\right)$$

and E_n can be otained by integration between T_c and T_n. This formulation allows $E(x)$ to be defined for a piecewise constant temperature profile. This integration is performed numerically using the trapezoid rule. The interface condition (4) is complex; therefore there are two real equations which can be satisfied by the choice of the parameters ω and Y_c. This can be done for each fixed value of $\alpha(\alpha = T_n/T_c)$ to obtain the stability curve.

THE NUMERICAL APPROXIMATION

To solve the differential equations (1), a routine, DEBDF, from the National Bureau of Standards subroutines library (CMLIB) is used.[3] This code was originally developed at the Sandia DOE laboratory. This is a solver intended for stiff equations. A non-stiff solver was also used and seemed to be somewhat less efficient. The differential equaution solver computes the functions $U_3(x, \omega, Y_c)$ and $U_4(x, \omega, Y_c)$. A routine from the NAG library, CO5NCF, is used to solve the system (3).[4] This routine was chosen because it does not require the values of the partial derivatives of U_3 and U_4 with respect to ω and Y_c. This routine starts with an initial guess for ω and Y_c and then iterates to find a solution for the system (3).

One of the input parameters to the differential equation solver is the error tolerance. The solver will attempt to integrate the equation with an estimated error less than this value. The experiments described here generally used a value of $1{\cdot}\mathrm{E}-7$ for this error tolerance. A similar error tolerance parameter must be input to the algebraic equation solver, CO5NCF. Here the value $1{\cdot}\mathrm{E}-4$ was generally used. When the solver converged, the final values of U_3 and U_4 were generally around $1{\cdot}\mathrm{E}-7$. When the error tolerance parameters for the DEBDF and CO5NCF routines are decreased from $1{\cdot}\mathrm{E}-6$ and $1{\cdot}\mathrm{E}-3$ to $1{\cdot}\mathrm{E}-8$ and $1{\cdot}\mathrm{E}-5$, the relative change in Y_c and ω for $\alpha = 35{\cdot}0$ is $2{\cdot}3\mathrm{E}-5$ and $1{\cdot}8\mathrm{E}-5$ respectivly. This is with the exponential temperature dependence described in Appendix II with $X_L = 0{\cdot}4$ and $X_H = 0{\cdot}6$.

Note that the stability curve is not single-valued. There may be two solutions for a single value of α; that is, there are two branches on the stability curve. The algebraic equation solver generally works well in the upper left region of the curve (see Figure 1). In this region it will converge to a solution even if the initial guess for (ω, Y_c) is rather poor. However, the initial guess has to be very good near the minimum of the curve, otherwise the equation solver will fail. Therefore, to obtain an initial guess, a quadratic extrapolation from the three previously computed values is used. For this purpose the stability curve is taken in log co-ordinates; that is, the curve consists of points ($\log\alpha$, $\log T_c$, $\log\omega$). The curve is parametrized by distance in this three-dimensional log space. If the three points are

$$(x_i, y_i, z_i) = (\log\alpha_i, \log T_{c,i}, \log\omega_i),$$

then the distance parameter is defined by

$$s_{i+1} - s_i = [(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2 + (z_{i+1} - z_i)^2]^{1/2}.$$

Each of the co-ordinates x_i, y_i and z_i is regarded as a function of s. Thus, to obtain an initial guess for the next point (x_4, y_4, z_4) from the three previously computed values, quadratic extrapolation with s as the independent variable is used. Once the initial guess is obtained, then the value of α is fixed by $\log\alpha = z_4$ and the algebraic solver is applied for this fixed value of α using (y_4, z_4) to determine the initial guess for Y_c and ω.

This extrapolation requires that a step size $s_4 - s_3$ be selected. This step size is made to depend on the curvature in the (α, Y_c) domain. The step size needs to be much smaller near the minimum of the curve in order that the initial guess be accurate enough to insure convergence of the algebraic solver. The curvature is approximated by $K = x'x'' - y'y''$, where $x = \log\alpha$ and $y = \log Y_c$ are regarded as functions of s. These derivatives with respect to s are obtained by differentiation of the

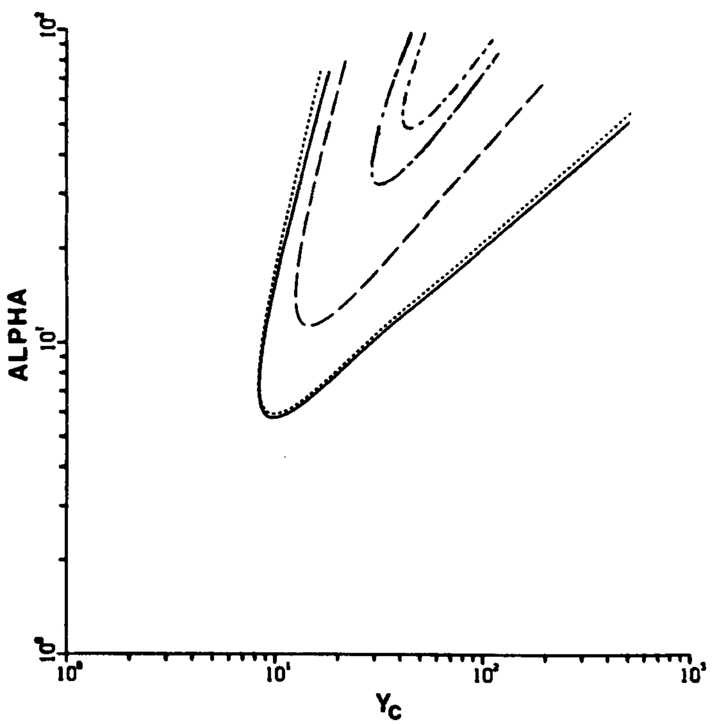

Figure 1. Neutral stability curves for temperature profiles of Figure 2: full curve, discontinuous temperature; dotted, $(X_L, X_H) = (0{\cdot}45, 0{\cdot}55)$; uniform broken, $(X_L, X_H) = (0{\cdot}, 1{\cdot}0)$; long–short broken $(X_L, X_H) = (-1{\cdot}0, 2{\cdot}0)$; medium–short broken, linear temperature (see 'Numerical results' section)

quadratic interpolating polynomial. The step size is given by $\Delta s = s_4 - s_3 = \varepsilon/\sqrt{|K|}$. In most cases we used $\varepsilon = 0{\cdot}04$. In addition, the step size is not allowed to exceed a maximum, which was usually taken as $\Delta s \leqslant 0{\cdot}15$.

A more general and systematic method for computing stability curves or surfaces in some parameter has been developed by Rheinboldt and Burkardt.[5] However, this method requires the computation of the Jacobian of the system of equation (3). This method is implemented in a subroutine package which is available from the Association for Computing Machinery (ACM).[5] We attempted to apply this method to the helium tube, using a finite difference approximation of the Jacobian. We were unable to get the method to converge. At this point we don't know the reason for the failure, but we suspect that the finite difference Jacobian may not have been accurate enough.

NUMERICAL RESULTS

The first numerical experiment was designed to check the method against the results obtained by Rott.[6] For this comparsion a piecewise constant temperature profile $(T_m(x))$ was used. The neutral curve for this case is the solid curve in Figure 2. This curve is in good agreement with the graph in Rott's paper. Reading from the graph in Rott's paper,[6] we find that the minimum value of Y_c on the stability curve is $8{\cdot}1 \pm 0{\cdot}4$. Our value is $8{\cdot}3$. The neutral curve depends on the sound speed $a(x)$, Prandtl number σ, viscosity temperature dependence parameter β, ratio of specific heats γ, tube length L, temperature $T_m(0)$, temperature ratio α and dimensionless ratio Y_c. The sound speed was

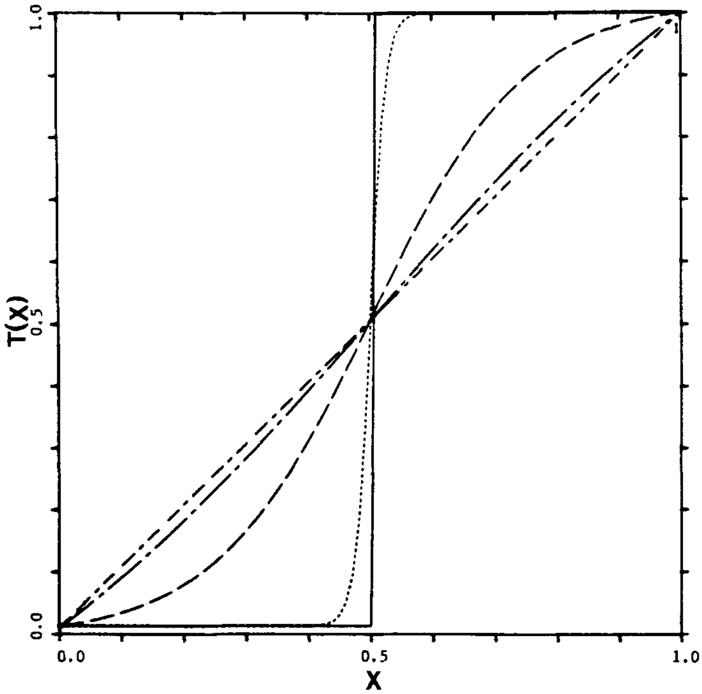

Figure 2. Normalized temperature profiles, $T_m(x)/T_m(L)$. Here $L = 1{\cdot}0\,\mathrm{m}$, $\beta = 0{\cdot}647$, $\sigma = 0{\cdot}667$, $\gamma = 1{\cdot}67$, $R_c = 2060\,\mathrm{J\,kg^{-1}\,K^{-1}}$

given by $a^2(x) = \gamma R_c T_m(x)$, with the constants given by $\gamma = 1{\cdot}67$, $R_c = 2060\,\mathrm{J\,kg^{-1}\,K^{-1}}$, $L = 1\,\mathrm{m}$, $T_m(0) = 4{\cdot}2\,\mathrm{K}$, $\beta = 0{\cdot}647$ and $\sigma = 0{\cdot}67$ (see Appendix I for the abbreviations of the units).

The neutral curves for five temperature profiles are plotted in the (Y_c, α) plane in Figure 1. The normalized temperature profiles, $T_m(x)/T_m(L)$, are shown in Figure 2. The first (solid) curve is obtained with a piecewise constant temperature profile. As discussed in the previous section, this case requires a different numerical method than the next four cases, which use the exponential temperature profile described in Appendix II. The second (dotted) curve is based on a smooth, but rapidly changing temperature profile where $(X_L, X_H) = (0{\cdot}45\,\mathrm{m}, 0{\cdot}55\,\mathrm{m})$. (See Appendix II for the definition of X_L and X_H.) The neutral curve for this case is almost the same as in the piecewise constant case. The broken curve in the centre has $(X_L, X_H) = (0, 1{\cdot}0\,\mathrm{m})$. The curve with the long–short chain pattern has $(X_L, X_H) = (-1{\cdot}0\,\mathrm{m}, 2{\cdot}0\,\mathrm{m})$. The last case is based on a linear temperature profile $T_m(x) = T_m(0) + x(\alpha - 1)T_m(0)$. The neutral curves can also be given in terms of the wave number ω or the tube radius r_0. This is shown in Figures 3 and 4. The region of stability is clearly larger if the temperature gradient along the tube is more uniform.

A plot of the real and imaginary parts of the eigenfunctions for the pressure is given in Figures 5 and 6. The plots are taken at a point near the minimum of the neutral curve for the piecewise constant (Figure 5) and the linear (Figure 6) temperature profiles.

As the temperature profiles became more nearly linear, there was some difficulty in starting the method. Unless the initial guess for the first root was within 10–30%, the root finder would fail. Therefore the temperature profile cannot be altered too rapidly in moving from case to case.

The curves could not be otained for values of Y_c greater than around 600 because the values of the Bessel functions overflowed the computer exponent range. However, these values were only needed to compute the ratios $f(x)$ and $f'(x)$ and these ratios were not too large. It should be possible to

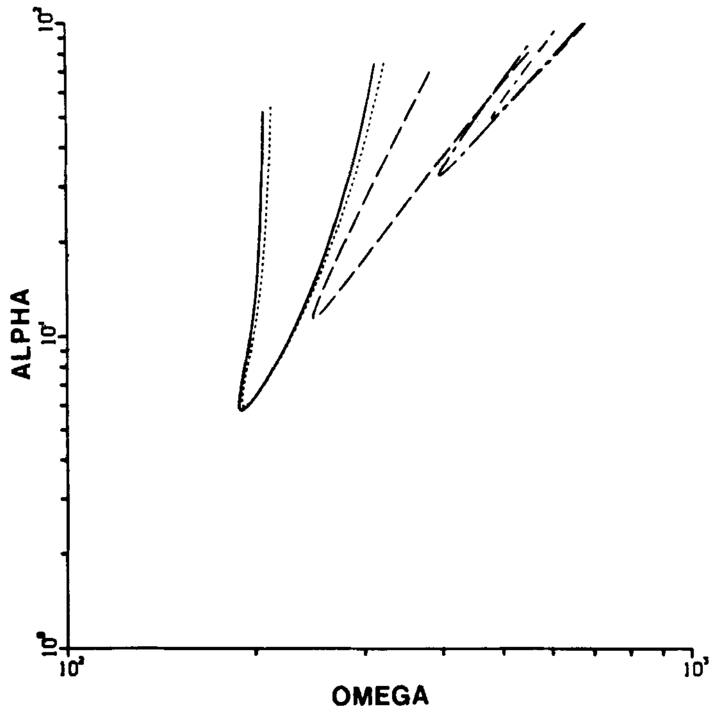

Figure 3. Neutral stability in terms of wave number ω

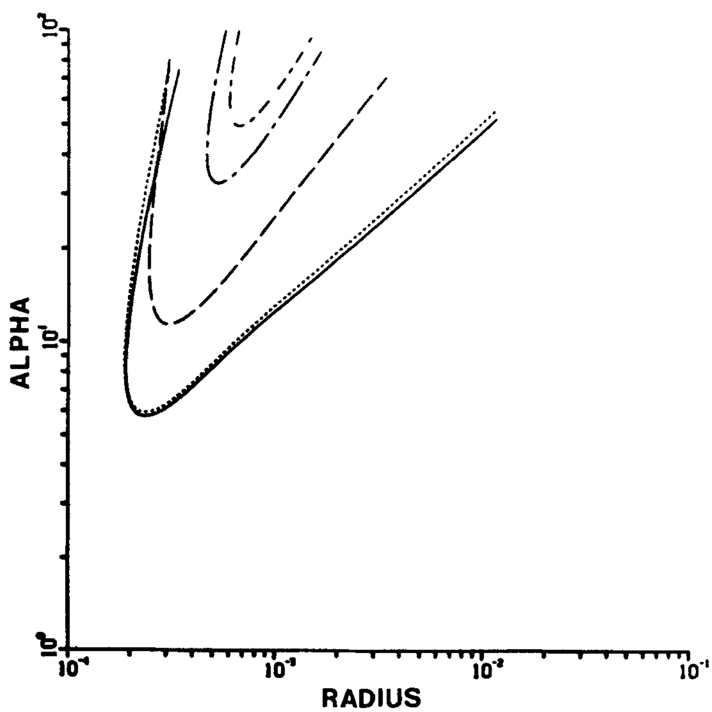

Figure 4. Neutral stability in terms of radius r_0

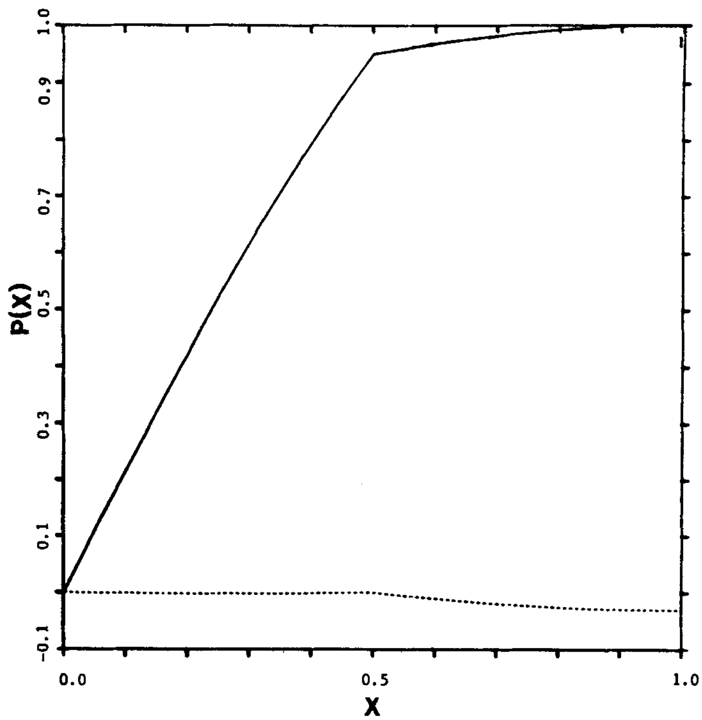

Figure 5. Real (full curve) and imaginary (dotted) part of pressure perturbation for piecewise constant temperature profile at $\alpha = 5{\cdot}9$, $Y_c = 9{\cdot}0$

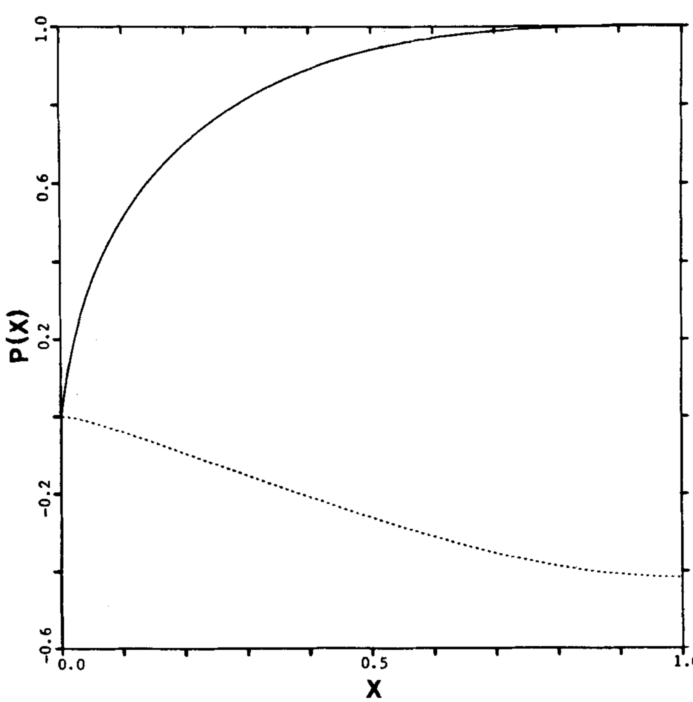

Figure 6. Real (full curve) and imaginary (dotted) part of pressure perturbation for linear temperature profile at $\alpha = 49$, $Y_c = 44$

compute the functions $f(x)$ and $f'(x)$ directly rather than using library routines to compute the Bessel functions. This would eliminate the overflow problem.

These computations were done on a CDC Cyber 840 computer. To find a single eigenvalue from a good initial guess requires about 12 s of central processor time. A neutral curve may require up to 50 points which takes about 10 min of computer time. This CDC computer can be represent numbers as large as 10^{300}. Some computers have an exponent range of 40 instead of 300. To run on such a computer, this algorithm would have to be modified to eliminate the exponent overflow in computing $f(x)$.

CONCLUSION

This paper describes a numerical method to compute the neutral curve for acoustic oscillations in a tube. This method allows fairly general temperature profiles for the gas contained in the tube. At present there are two versions of the code; one assumes a piecewise constant temperature profile, the other assumes a smooth (i.e., differentiable) monotone temperature profile. The present algorithm, assumes a Prandtl number with no temperature dependence and assumes the sound speed has an ideal gas dependence on temperature.

The computational results show that a case with a uniform temperature gradient along the tube is more stable than a case where the temperature suddenly jumps from the cold to the hot end value.

It should be possible to extend the algorithm to represent more accurately a non-ideal gas and also to allow a tube whose radius varied with the distance along it. Also, the algorithm should be modified so that the computations required to evaluate the function $f(x)$ do not need such a large exponent range.

ACKNOWLEDGEMENTS

We wish to thank Dr. Ray Radebaugh, National Bureau of Standards, for suggesting this problem and providing assistance towards its solution.

This work was supported in part by the National Aeronautics and Space Administration.

APPENDIX I. NOTATION AND UNITS

Symbol	*Units**	*Explanation*
α	—	$T_m(L) T_m(0)$ temperature ratio
β	—	Parameter for viscosity temperature dependence, $\nu(T) = CT^{(1+\beta)/2}$, $\beta = 0{\cdot}647$
γ	—	Ratio of specific heats
ν	$m^2 s^{-1}$	Kinematic viscosity
σ	—	$\sigma = \mu c_p/k$ = Prandtl number
θ	—	$T'_m(x)/T_m(x)$
a	$m s^{-1}$	Sound speed
i	—	$\sqrt{(-1)}$
j	m	Location of temperature jump
$J_k(z)$	—	Bessel function of order k
k	$J m^{-1} s^{-1} K^{-1}$	Thermal conductivity in gas
L	m	Tube length
p	Pa	Pressure perturbation

* m = metres, K = degrees Kelvin, J = Joule, s = seconds, Pa = pascals.

r_0	m	Tube radius
T_c	K	Temperature at cold end of tube, $T_c = T_m(0)$
T_m	K	Mean temperature function
T_n	K	Temperature at hot end of tube, $T_n = T_m(L)$
x	m	Axial distance along tube
ω	s^{-1}	Wave number of oscillation
Y_c	—	$Y_c = [\omega/\nu(0)]^{1/2} r_0$ = dimensionless boundary layer parameter

APPENDIX II. THE TEMPERATURE PROFILE

The mean temperature function, $T_m(x)$, is designed to be a monotone function which increases from a value T_c at $x = 0$ to a value $T_n = \alpha T_c$ at $x = L$. The value T_c is always taken to be the temperature of liquid helium at atmospheric pressure, namely 4.2 K. The function is smooth; that is, it possesses derivatives of all orders. The function requires, in addition to T_c and α, two more parameters for its definition. These parameters are the end points X_L and X_H of the interval over which the function undergoes most of its increase from T_c to T_n. Thus the temperature $T_m(X_L)$ is only slightly greater than T_c and $T_m(X_H)$ is only slightly less than T_n.

The definition of T_m follows:

$$T_m(x) = (T_0 e^y + T_1 e^{-y})/(e^y + e^{-y}),$$

$$y = s(x - X_M), \quad s = 4/(X_H - X_L), \quad X_M = (X_H + X_L)/2,$$

$$T_0 = (\beta_H T_L - \beta_L T_H)/(A_L \beta_H - A_H \beta_L),$$

$$T_1 = (A_L T_H - A_H T_L)/(A_L \beta_H - A_H \beta_L),$$

$$A_L = E_1^{-1}/(E_1 + E_1^{-1}), \quad \beta_L = E_1/(E_1 + E_1^{-1}), \quad E_1 = e^{sX_M}$$

$$A_H = E_2/(E_2 + E_2^{-1}), \quad \beta_H = E_2^{-1}/(E_2 - E_2^{-1}), \quad E_x = e^{s(L - X_M)}.$$

REFERENCES

1. E. Tward and P. V. Mason, 'Damping of thermoacoustic oscillations', *Adv. Cryog. Eng.*, **27**, 807–815 (1982).
2. N. Rott, 'Damped and thermally driven oscillations in wide and narrow tubes', *J. Appl. Math. Phys.*, **20**, 230–243 (1969).
3. *Guide to Mathematical Software, NBSIR 84-2824*, National Bureau of Standards, Scientific Computing Division, Washington, DC 20234 (1984).
4. *NAG Fortran Library Manual*, Numerical Algorithms Group Ltd, 7 Banbury Road, Oxford OX2 6NN.
5. W. Rheinboldt and J. Burkardt, 'A locally parametrized continuation process', *ACM Trans. Math. Software*, **9** (2), 215–235 (1983).
6. N. Rott, 'Thermally driven acoustic oscillations. Part II. Stability limit for helium', *J. Appl. Math. Phys.*, **24**, 54–72 (1973).